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Note

Metric Projection onto a Lattice in L_1

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In this note it is shown that the L_1 metric projection onto a lattice is Lipschitz continuous, and that it has a Lipschitz continuous selection.

In this note it is shown that the L_1 metric projection onto a lattice is Lipschitz continuous, and that it has a Lipschitz continuous selection. Our results apply, in particular, to the theories of isotonic regression and conditional medians given a σ-algebra (the L_1 version of conditional expectation), so this work may prove useful to probabilists as well as approximation theorists.

The cornerstone of the existential theory of selections is Michael's Selection Theorem [8], which says, in essence, that the set, $H(X)$, of closed, bounded, convex, nonempty subsets of a normed linear space X has a continuous selection, i.e., a continuous choice function, $\Psi: H(X) \to X$. Suppose X has infinite dimension. In this case it has been shown that there can be no Lipschitz continuous choice function [11]. If $\mathscr{K}$ is a proximinal subset of X, then the metric projection of X onto $\mathscr{K}$ has for its range a subset of $H(X)$, and a selection restricted to this subset (and composed with the metric projection) is called a *metric selection*. In [3], several characterizations were given of proximal subspaces which admit Lipschitz continuous selections. The present note describes a class of such subspaces.

When $\dim(X) < \infty$, the Steiner point has been shown to be a Lipschitz continuous selection [10]. However, Vitale [14] showed that the Steiner

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point cannot be extended continuously to all convex bodies in an infinite dimensional space. The Steiner point is the only constructive approach to the Lipschitz continuous selection problem known to the authors. On the other hand, there is good news regarding the *metric* selection. Ubhaya [13] described a large family of approximating subsets of L_∞ each of which has a distance-reducing metric selection. Ubhaya's selection is calculable via an elementary characterization. The present study is primarily based in the less-studied space, L_1. Our family of approximating sets is similar to, but less general than, Ubhaya's.

Now we will describe the context in which we are working. If $(\Omega, \mathscr{A}, \mu)$ is a measure space and $1 \leqslant p < \infty$, let $L_p := L_p(\Omega, \mathscr{A}, \mu)$ consist of all μ-measurable functions $f:\Omega \to \mathbf{R}$ such that $\|f\|_p := \{\int_\Omega |f|^p\}^{1/p}$ and let L_∞ consist of all functions $f: \Omega \to \mathbf{R}$ such that $\|f\|_\infty := \sup_{x\in\Omega} |f(x)| < \infty$. If $p \in [1, \infty]$, $\mathscr{K} \subset L_p$, and $f \in L_p$, we say that g is a *best L_p-approximant* to f from $\mathscr{K}$ if $g \in \mathscr{K}$ and

$$\|f-g\|_p = \inf_{h\in\mathscr{K}} \|f-h\|_p.$$

For each $p \in [0, \infty]$, we define a set-valued function Π_p by requiring that for each $f \in L_p$, $\Pi_p(f)$ shall consist of all best L_p-approximants to f from $\mathscr{K}$. We call Π_p the ($\|\cdot\|_p$) *metric projection* of L_p onto $\mathscr{K}$. Let $L := L_1 \cap L_\infty$. Let Π denote the restriction to L of Π_1.

For the study of the continuity of Π, we topologize its codomain using the *Hausdorff distance*: If A and B are nonempty subsets of a metric space (M, d), let

$$H_d(A, B) := \max\left\{\sup_{a\in A}\inf_{b\in B} d(a, b), \sup_{b\in B}\inf_{a\in A} d(a, b)\right\}.$$

If d is the metric induced by $\|\cdot\|_1$ (respectively, $\|\cdot\|_\infty$) we will denote H_d by H_1 (resp., H_∞). In [7], Legg and Townsend gave an example to show that $\|f_n - f\|_1 \to 0$ does not always imply that $H_1(\Pi(f_n), \Pi(f)) \to 0$, but they proved that, when $\Omega = [0, 1]$, μ is Lebesgue measure, and $\mathscr{K}$ consists of all nondecreasing functions in L_1, then, for every $f, f' \in L_1$, $H_\infty(\Pi(f), \Pi(f')) < 8\,\|f-f'\|_\infty$. We will now show that the Legg–Townsend theorem is true in a much more general context, and that Π is, in fact, distance reducing, i.e., the Lipschitz constant can be lowered from eight to one.

If $\{f_\lambda : \lambda \in \Lambda\} \subset L_p$, let $\bigvee_{\lambda\in\Lambda} f_\lambda := \sup_{\lambda\in\Lambda} f_\lambda$, and $\bigwedge_{\lambda\in\Lambda} f_\lambda := \inf_{\lambda\in\Lambda} f_\lambda$. If $\Lambda = \{1, 2\}$, we denote $\bigvee_{\lambda\in\Lambda} f_\lambda$ by $f_1 \vee f_2$, and $\bigwedge_{\lambda\in\Lambda} f_\lambda$ by $f_1 \wedge f_2$. We say that a subset $\mathscr{K}$ of L_p is a *lattice* (respectively, a *σ-complete lattice*) if $\bigvee_{\lambda\in\Lambda} f_\lambda, \bigwedge_{\lambda\in\Lambda} f_\lambda \in \mathscr{K}$ whenever $\{f_\lambda : \lambda \in \Lambda\} \in \mathscr{K}$ and Λ is finite (resp., at most countable). If $\mathscr{K} \subset L_p$ and $\mathscr{K} + c = \mathscr{K}$ for every $c \in R$ (i.e., $\mathscr{K}$ is invariant under translation by constants), then it follows readily from the definitions that Π_p is additive modulo the set of all constant functions, i.e.,

$$\Pi_p(f+c)=\Pi_p(f)+c \tag{1}$$

for every $f\in L_p$ and $c\in R$. The following theorem states that the L_1 metric projection is $\|\cdot\|_\infty$ Lipschitz continuous.

THEOREM 1. *Suppose that* $\mathcal{K}\subset L$ *is an* $\|\cdot\|_1$*-closed,* σ*-complete lattice which is invariant under translation by constants. If* $f, f'\in L_\infty$*, then*

$$H_\infty(\Pi(f), \Pi(f'))\leqslant \|f-f'\|_\infty.$$

Furthermore, the estimate is sharp.

Proof. Let $\varepsilon:=\|f-f'\|_\infty$. By Theorem 4 in [4], neither $\Pi(f)$ nor $\Pi(f')$ is empty. Choose any $h_0\in\Pi(f)$ and $g_0\in\Pi(f')$. Since $f'-\varepsilon<f<f'+\varepsilon$ and $\Pi(f'\pm\varepsilon)=\Pi(f')\pm\varepsilon$, Theorem 18 in [4] implies that both $g_1:=[h_0\wedge(g_0-\varepsilon)]+\varepsilon$ and $g_2:=[h_0\vee(g_0+\varepsilon)]-\varepsilon$ are elements of $\Pi(f')$. Let $g^*:=g_1\vee(h_0-\varepsilon)$. We claim that $g^*\in\Pi(f')$. Suppose not. Let $A:=[h_0<g_0-\varepsilon]:=\{x\in\Omega: h_0(x)<g_0(x)-\varepsilon\}$, $B:=[g_0-\varepsilon\leqslant h_0\leqslant g_0+\varepsilon]$, and $C:=[g_0+\varepsilon<h_0]$. Proposition 2.1.2 in [1] implies that each of A, B, and C is in $\mathcal{A}$. Note that

$$\|f'-g\|_1=\int_A|f'-g|+\int_B|f'-g|+\int_C|f'-g|$$

for every $g\in\mathcal{K}$. Since $g^*=g_1$ on $A\cup B$ and $g_1\in\Pi(f')$, it must be that

$$\int_C|f'-g_2|=\int_C|f'-g^*|>\int_C|f'-g_1|=\int_C|f'-g_0|.$$

But $g_2=g_0$ on $A\cup B$, so $\|f'-g_2\|_1>\|f'-g_0\|_1$, a contradiction. This shows that $g^*\in\Pi(f')$. Clearly, $\|g^*-h_0\|_\infty\leqslant\varepsilon$. The construction of an $h^*\in\Pi(f)$ such that $\|h^*-g_0\|_\infty<\varepsilon$ is symmetric.

To see that the estimate is sharp, choose $f\in\mathcal{K}$ and let $f':=f+\varepsilon$. Then $\Pi(f)=\{f\}$ and $\Pi(f')=\{f+\varepsilon\}$, so $H_\infty(\Pi(f),\Pi(f'))=\varepsilon=\|f-f'\|_\infty$. This concludes the proof of Theorem 1.

We now describe a context in which Π has a $\|\cdot\|_\infty$-distance reducing selection. Suppose that, in addition to the above assumptions, $\mathcal{K}$ is $\|\cdot\|_\infty$-boundedly compact and μ is finite. Then, for every $p\in(1,\infty)$, $\mathcal{K}$ is a $\|\cdot\|_p$-closed convex subset of the uniformly convex Banach space, L_p, so Π_p is single valued. In this case, for each $f\in L_p$, we will denote the single element of $\Pi_p(f)$ by $\pi_p(f)$. In [5], it was shown that each $f \in L_\infty$ has a distinguished best L_1-approximant, $m_1(f)$, from $\mathscr{K}$ with the property

$$\|\pi_p(f) - m_1(f)\|_1 \to 0 \qquad \text{as} \quad p \downarrow 1. \tag{2}$$

The function $m_1(f)$ is called the *natural* best $\|\cdot\|_1$ approximant to f from $\mathscr{K}$, so we call the operator $f \mapsto m_1(f|\mathscr{K})$ the *natural selection*. An internal characterization of $m_1(f)$ was also given in [5], viz., $\Phi(m_1(f)) \leqslant \Phi(g)$, $g \in \Pi(f)$, where $\Phi(h) := \int |f-h| \ln |f-h|$.

THEOREM 2. *Suppose that μ is finite and $\mathscr{K} \subset L_\infty$ is an $\|\cdot\|_1$-closed and $\|\cdot\|_\infty$-boundedly compact lattice which is invariant under translation by constants. If $f, f' \in L_\infty$, then*

$$\|m_1(f) - m_1(f')\|_\infty \leqslant \|f - f'\|_\infty.$$

Furthermore, the estimate is sharp.

Proof. By (3ii) in [6], for any $p \in (1, \infty)$ and $u, v \in L_p$,

$$\pi_p(u) \leqslant \pi_p(v) \qquad \text{whenever} \quad u \leqslant v. \tag{3}$$

Since μ is finite, $L_\infty \subset L_p$, so (1) and (3) imply that for any $p \in (1, \infty)$ and $u, v \in L_\infty$,

$$\|\pi_p(u) - \pi_p(v)\|_\infty \leqslant \|u - v\|_\infty. \tag{4}$$

Let $f \in L_\infty$ and $g_0 \in \mathscr{K}$ be fixed. By (4), $\|\pi_p(f)\|_\infty \leqslant \|f - g_0\|_\infty + \|g_0\|_\infty$, i.e., the net $\{\pi_p(f) : p \in (1, \infty)\}$ is uniformly bounded. Since $\mathscr{K}$ is boundedly compact, there exist $p_n \downarrow 1$ and $g \in \mathscr{K}$ such that $\|\beta_n\|_\infty \to 0$, where $\beta_n := \pi_{p_n}(f) - g$. Since $\|\beta_n\|_1 \leqslant \mu(\Omega)\, \|\beta_n\|_\infty$, (2) implies that $g = m_1(f)$ a.e. By similar reasoning, given $f' \in L_\infty$, the sequence $\{\pi_{p_n}(f')\}$ is uniformly bounded so it contains a subsequence $\{\pi_{q_n}(f')\}$ which converges uniformly to $m_1(f')$. The triangle inequality, and (4), can now be used to establish the estimate. That the estimate is sharp is shown exactly as in Theorem 1. This concludes the proof of Theorem 2.

We conclude with a discussion of some contexts in which the above theorems are applicable. Let $\mathscr{B}$ consist of the empty set and all sets of the form $\{(a, 1], (a, 1] : 0 \leqslant a < 1\}$. Then $\mathscr{B}$ is a σ-lattice of subsets of $[0, 1]$, and the set, $\mathscr{K}$, of all L_∞ functions measurable with respect to $\mathscr{B}$ satisfies the hypotheses of Theorem 2. Bounded compactness follows from an obvious extension of VIII. 4.2 in [9]. This set consists of all nondecreasing functions on $[0, 1]$, so our theory includes that of Legg and Townsend. Approximation by nondecreasing functions is called *isotonic regression* by statisticians [12]. The continuity of the L_∞ metric projection onto this set will be discussed in a subsequent publication.

Suppose $\mathscr{K}$ satisfies the hypothesis of one of our theorems. For $i=1, 2, ..., n$, let Ω_i be the set of ordered pairs $\{(x, i) : x \in \Omega\}$; let $\Omega^* := \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_n$; let $\mathscr{A}^*$ consist of all sets of the form $\{(x, i) : x \in A_i \in \mathscr{A},\ i = 1, 2, ..., n\}$; for $A = A_1 \cup A_2 \cup \cdots A_n \in \mathscr{A}^*$, let $\mu^*(A) := \sum_{i=1}^n \mu\{x : (x, i) \in A_i\}$; and let $\mathscr{K}^*$ consist of all functions $y\colon \Omega^* \to R$ such that $g(x, i) = g(x, j)$ for $i \neq j$ and the function $h(x) := g(x, i)$ is in $\mathscr{K}$. Then $\mathscr{K}^*$ satisfies the hypothesis of our theorem (with $(\Omega, \mathscr{A}, \mu) = (\Omega^*, \mathscr{A}^*, \mu^*)$). If $f_1, ..., f_n \in L_1$ define $f^*\colon \Omega^* \to R$ by $f^*(x, i) = f_i(x)$. The approximation of f^* by elements of $\mathscr{K}^*$ is equivalent to the simultaneous approximation of $f_1, f_2, ..., f_n$ by elements of $\mathscr{K}$, using the "sum" norm. In this norm, the measure of deviation of $(f_1, f_2, ..., f_n)$ from $h \in \mathscr{K}$ is $\sum_{i=1}^n \|f_i - h\|_1$. (This example was our original motivation for the consideration of lattices of functions.)

If $\mathscr{B}$ is a sub σ-algebra of $\mathscr{A}$ and $\mathscr{K}$ consists of all functions in L_1 which are measurable with respect to $\mathscr{B}$, then, by Section 2.1 in [1], $\mathscr{K}$ satisfies the hypothesis of Theorem 1. In this case, the elements of $\Pi(f)$ are known as *conditional medians* of f. Darst [2] showed that $f_\infty := \lim_{p \to \infty} \pi_p(f)$ always exists, so an argument similar to that of Theorem 2 shows that the operator $f \mapsto f_\infty$ is a distance-reducing metric selection. It would be of interest to explore the relationship between this selection and that described by Ubhaya [13].

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